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The geometry of pure spinor space

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Abstract: We investigate the complex geometry of $D = 10$ pure spinor space. The Kähler structure and the corresponding metric giving rise to the desired Calabi–Yau property are determined, and an explicit covariant expression for the Laplacian is given. The metric is not that of a cône obtained by embedding pure spinor space in a flat space of unconstrained spinors. Some directions for future studies, concerning regularisation and generalisation to eleven dimensions, are briefly discussed.

1. INTRODUCTION AND BACKGROUND

As is well known, pure spinor superfield formalism solves the problem of giving off-shell supersymmetric formulations of maximally supersymmetric field theories and string theories (see *e.g.* refs. [1-17]). It allows in principle for the covariant calculation of amplitudes in such theories, see refs. [18-22] and references therein.

Some problems remain, though. One of the main issues is how to make integrals over the pure spinor space convergent. The pure spinor space is non-compact and has disconnected boundary components at the origin and at infinity. Divergences at infinity are easily dealt with [18]. Divergences at the origin are related to the question of gauge fixing. Normally, one constructs a gauge fixing operator b (the “ b ghost”) as a differential operator and demands that $b\Psi = 0$ (Ψ being the pure spinor superfield). This operator [18], in its simplest version, is singular at the origin $\lambda = 0$ of pure spinor space, and a sufficient number of propagators containing b will generate divergences at the origin (while at the same time fermionic integrals become over-saturated). A schematic, but technically quite complicated, recipe for the regularisation of such divergences at higher loops was given in ref. [20]. This method demonstrates in principle the existence of a regular b operator, but can hardly be said to provide an explicit form suited for calculations.

We are convinced that the work with trying to resolve these issues will benefit from a clearer understanding of the geometry of the pure spinor space, and this is the purpose of the present paper. The questions above concerning gauge fixing and regularity will not be solved here, but we hope this can provide a starting point. We comment on this in the final section.

Many of the statements made here concerning pure spinor geometry are already well known, but are included for sake of completeness. See *e.g.* refs. [18,20,23-27] for some geometrical considerations. The main results of the present paper is the form of the Kähler potential and the metric leading to the (necessary) Calabi–Yau property, together with the covariant construction of metric-dependent differential operators, such as the Laplacian.

2. PROPERTIES OF PURE SPINOR SPACE

The pure spinor constraint $(\lambda\gamma^a\lambda) = 0$ determines an 11-dimensional complex space, holomorphically embedded in the 16-dimensional space of unconstrained spinors. In the non-minimal formalism [18], where integration is well defined, one also considers the complex conjugate spinor $\bar{\lambda}_\alpha$ (of course fulfilling $(\bar{\lambda}\gamma^a\bar{\lambda}) = 0$) and the fermionic variable r_α , with

$(\bar{\lambda}\gamma^a r) = 0$. Due to the latter constraint, the variables r_α may be identified with $d\bar{\lambda}_\alpha$. The non-minimal BRST operator is

$$Q = (\lambda D) + (r \frac{\partial}{\partial \bar{\lambda}}) = (\lambda D) + \bar{\partial} , \quad (2.1)$$

where $\bar{\partial} = d\bar{\lambda}_\alpha \frac{\partial}{\partial \bar{\lambda}_\alpha}$ is the usual Dolbeault operator [20] and D_α the fermionic covariant derivative.

Letting a field Ψ depend also on $\bar{\lambda}$ and r means letting it be a cochain with antiholomorphic indices. An action may be written as

$$S = \int \Omega \wedge (\Psi \wedge Q\Psi + \dots) \quad (2.2)$$

(where integration over superspace coordinates x and θ has been suppressed for brevity). Ω here denotes a holomorphic top form, which is described in section 2.2.

2.1. COORDINATES ON PURE SPINOR SPACE

The stability group for a pure spinor is $SU(5)$. Spinors of the two chiralities, **16** and $\overline{\mathbf{16}}$ decompose under $SU(5) \times U(1) \subset Spin(16)$ as

$$\begin{aligned} \mathbf{16} &\rightarrow \mathbf{1}_{-5/2} \oplus \mathbf{10}_{-1/2} \oplus \bar{\mathbf{5}}_{3/2} , \\ \overline{\mathbf{16}} &\rightarrow \mathbf{5}_{-3/2} \oplus \overline{\mathbf{10}}_{1/2} \oplus \mathbf{1}_{5/2} . \end{aligned} \quad (2.3)$$

Consider a chiral spinor Λ in **16** as an even form,

$$\Lambda = \Lambda_0 + \Lambda_2 + \Lambda_4 . \quad (2.4)$$

We will write $\ell \equiv \Lambda_0$. The γ -matrices act as $\omega \wedge \Lambda$ and $\iota_v \Lambda$, where ω is a 1-form and v a vector, and this extends also to the odd forms, containing a spinor of opposite chirality. The $so(10)$ generators outside $su(5) \oplus u(1)$ are given as $\mu \wedge$ and ι_m , where μ is a 2-form and m an antisymmetric bivector. The invariant product of **16** and $\overline{\mathbf{16}}$ is

$$\langle \Lambda, K \rangle = \star(\Lambda_0 K_5 - \Lambda_2 \wedge K_3 + \Lambda_4 \wedge K_1) . \quad (2.5)$$

The pure spinor constraint $(\lambda\gamma^a\lambda) = 0$ becomes

$$\langle \Lambda, \omega \wedge \Lambda \rangle = 0 = \langle \Lambda, \iota_v \Lambda \rangle . \quad (2.6)$$

The first of these conditions imply that $\Lambda_4 = \frac{1}{2}\ell^{-1}\Lambda_2 \wedge \Lambda_2$ (in the patch where $\ell \neq 0$). Once it is satisfied, $\langle \Lambda, \iota_v \Lambda \rangle \sim \ell^{-1}\star(\iota_v \Lambda_2 \wedge \Lambda_2 \wedge \Lambda_2) = 0$. The 11-dimensional space of $D = 10$ pure spinors is parametrised by ℓ and Λ_2 .

2.2. KÄHLER STRUCTURE AND THE HOLOMORPHIC TOP FORM

There is an $so(10)$ -invariant $(11, 0)$ -form [20] on pure spinor space,

$$\Omega = \ell^{-3} d\lambda d^{10}\Lambda_2 . \quad (2.7)$$

It is clear from above that its $U(1)$ charge is 0, and it remains to show the invariance under $\mu\wedge$ and ι_m as above. The first of these is trivial, since it does not pull down any non-linear contribution from Λ_4 . Under ι_m we have

$$\begin{aligned} \delta_m \ell &= \iota_m \Lambda_2 , \\ \delta_m \Lambda_2 &= \iota_m \Lambda_4 = \frac{1}{2}\ell^{-1}\iota_m(\Lambda_2 \wedge \Lambda_2) . \end{aligned} \quad (2.8)$$

Using the second equation, together with $d\Lambda_{AB}(d^9\Lambda)^{CD} = \frac{1}{10}\delta_{AB}^{CD}d^{10}\Lambda_2$ where $(d^9\Lambda_2)^{AB}$ is defined by $d^{10}\Lambda_2 \equiv d\Lambda_{AB}(d^9\Lambda)^{AB}$, a short calculation shows that the volume form (2.7) is invariant under $so(10)$ precisely when the power of ℓ in the prefactor is -3 . The factor ℓ^{-3} of course reflects the same power of λ in the covariant "measure", given as

$$\Omega \propto (\lambda\bar{\lambda})^{-3} \bar{\lambda}_{\alpha_1} \bar{\lambda}_{\alpha_2} \bar{\lambda}_{\alpha_3} T^{\alpha_1\alpha_2\alpha_3}_{\beta_1\dots\beta_{11}} d\lambda^{\beta_1} \wedge \dots d\lambda^{\beta_{11}} , \quad (2.9)$$

T being the unique $so(10)$ -invariant tensor with 3 pure spinor indices and 11 antisymmetric cospinor indices (*i.e.*, Clebsch–Gordan coefficients for the formation of a singlet from $(00030) \otimes (\wedge^{11}(00001))$). It was demonstrated in ref. [28] that Ω of eq. (2.9) is independent of $\bar{\lambda}$, *i.e.*, that $\bar{\partial}\Omega = 0$. The $so(10)$ invariance ensures that Ω of eq. (2.7) is globally defined. The existence of such an Ω is the Calabi–Yau property.

The existence of the holomorphic top form is essential for the gauge invariance of the action (2.2), since Q can not be partially integrated if not $\bar{\partial}\Omega = 0$. Equally important is the (obvious) non-exactness of Ω , $\Omega \neq \bar{\partial}\xi$. In the field Ψ , all cohomology has minimal representatives independent of $\bar{\lambda}$ and r , *i.e.*, holomorphic $(0, 0)$ -forms. For the action not to

produce an unwanted doubling of components fields, the anti-holomorphic top form $\bar{\Omega}$ must not represent $\bar{\partial}$ cohomology, and thus $\bar{\Omega} = \bar{\partial}\xi_{(0,10)}$. Such a form $\xi_{(0,10)}$ is readily constructed as

$$\xi_{(0,10)} \propto (\lambda\bar{\lambda})^{-3} \lambda^{\alpha_1} \lambda^{\alpha_2} \lambda^{\alpha_3} \bar{T}_{\alpha_1\alpha_2\alpha_3}^{\beta_1\dots\beta_{11}} \bar{\lambda}_{\beta_1} d\bar{\lambda}_{\beta_2} \wedge \dots d\bar{\lambda}_{\beta_{11}} . \quad (2.10)$$

This asymmetry in the $\bar{\partial}$ cohomology is possible since the pure spinor space is non-compact. Also the volume form $\text{Vol} = \Omega \wedge \bar{\Omega}$ is cohomologically trivial. This does not mean that the “volume” (which needs regularisation at infinity) vanishes, since there are two boundary components, at zero and infinity. The Kähler potential is globally defined, so neither does the Kähler form represent cohomology, which is consistent with the triviality of the volume form.

The full complex 16 \mathbb{C} -dimensional spinor space allows a Kähler structure (in fact, infinitely many). The flat geometry corresponds to the Kähler potential $K_0 = (\lambda\bar{\lambda})$. Contrary to what is sometimes assumed, this will *not* be the actual geometry, but we will nevertheless examine the induced geometry on the pure spinors. We use the coordinates $(z^m; \bar{z}^{\bar{m}}) = (\ell, \Lambda_{ab}; \bar{\ell}, \bar{\Lambda}^{ab})$ from above. The Kähler potential for the induced geometry is inherited from the embedding space:

$$K_0 = (\lambda\bar{\lambda}) = \langle \Lambda, \bar{\Lambda} \rangle = \ell\bar{\ell} - \frac{1}{2}\text{tr}(\Lambda\bar{\Lambda}) - \frac{1}{4}(\ell\bar{\ell})^{-1}X , \quad (2.11)$$

where $X = \text{tr}(\Lambda\bar{\Lambda})^2 - \frac{1}{2}(\text{tr}(\Lambda\bar{\Lambda}))^2$, and the Kähler form is given as $\omega_0 = h_{m\bar{n}} dz^m \wedge d\bar{z}^{\bar{n}} = \partial\bar{\partial}K_0$ (we use “ h ” for this metric and reserve “ g ” for later). The explicit form of the metric is explored in the following subsection. The volume form is $\text{Vol}_0 = \sqrt{h} d^{11}z d^{11}\bar{z} \propto \wedge^{11}\omega_0$. Now, since K_0 is homogeneous of degree (1,1) in the coordinates, it is obvious that the metric, and thus also its determinant, is homogeneous of degree (0,0). A calculation using Mathematica shows that indeed

$$\sqrt{h} = \frac{(\lambda\bar{\lambda})^3}{(\ell\bar{\ell})^3} . \quad (2.12)$$

This is not acceptable — as argued in the previous subsection it is crucial that there is a holomorphic (11,0)-form Ω such that $\text{Vol} \propto \Omega \wedge \bar{\Omega}$, *i.e.*, that \sqrt{g} factorises as $\sqrt{g} = f\bar{f}$, where $f(z)$ is holomorphic.

The only Lorentz invariant that does not vanish on the pure spinor space is $(\lambda\bar{\lambda})$. Therefore, a Kähler potential must be taken as a function of this invariant. If we take $K = \frac{11}{8}K_0^{8/11} = \frac{11}{8}(\lambda\bar{\lambda})^{8/11}$, the metric will be homogeneous in both z and \bar{z} of degree $-\frac{3}{11}$, and \sqrt{g} will be of degree -3 . Again, using Mathematica, we see that $\sqrt{g} = \frac{8}{11}(\ell\bar{\ell})^{-3}$, so the invariant holomorphic volume form (2.7) is reproduced, and $\text{Vol} \propto \wedge^{11}\omega \propto \Omega \wedge \bar{\Omega}$. More generally, a Kähler potential $K_p = p^{-1}(\lambda\bar{\lambda})^p$ gives $\sqrt{g_p} = p(\lambda\bar{\lambda})^{11p-8}(\ell\bar{\ell})^{-3}$.

So the metric on the full spinor space giving the induced metric is

$$ds^2 = (\lambda\bar{\lambda})^{-3/11} \left(d\lambda d\bar{\lambda} - \frac{3}{11} (\lambda\bar{\lambda})^{-1} (d\lambda\bar{\lambda})(\lambda d\bar{\lambda}) \right) , \quad (2.13)$$

obtained from the Kähler potential $K = \frac{11}{8} (\lambda\bar{\lambda})^{8/11}$. Pure spinor space should not be thought of as a cône embedded in flat space, but in a space with a metric which is already highly singular at the origin.

This makes the pure spinor space Ricci flat. On Kähler manifolds, the Ricci form (related to the Ricci tensor the same way as the Kähler form to the metric) is given by $\varrho = \partial\bar{\partial} \log \sqrt{g}$, which vanishes when \sqrt{g} factorises as $\sqrt{g} = f\bar{f}$, as above. With $K_p = p^{-1}(\lambda\bar{\lambda})^p$, one gets $\varrho = (11p - 8)\partial\bar{\partial} \log(\lambda\bar{\lambda})$. This vanishes only when $p = \frac{8}{11}$ and has rank 10 for other values of p (its components along λ and $\bar{\lambda}$ vanish). The scalar curvature is $R = 20(11p - 8)(\lambda\bar{\lambda})^{-1}$.

2.3. THE EXPLICIT METRIC

The metric in the system with coordinates (ℓ, Λ_{AB}) defined above can be given by using the pure spinor constraint in the expression for the metric (or, the Kähler form) of the embedding space. Let us first do this for the flat embedding space metric with Kähler potential $K_0 = (\lambda\bar{\lambda})$. We get

$$\begin{aligned} h^{0\bar{0}} &= 1 - \frac{1}{4}(\ell\bar{\ell})^{-2}X , \\ h^{AB,0} &= -(\ell\bar{\ell})^{-2}\ell \left[(\bar{\Lambda}\Lambda\bar{\Lambda})^{AB} - \frac{1}{2}\bar{\Lambda}^{AB}\text{tr}(\Lambda\bar{\Lambda}) \right] , \\ h^{AB,\bar{C}\bar{D}} &= 2 \left[1 - \frac{1}{2}(\ell\bar{\ell})^{-1}\text{tr}(\Lambda\bar{\Lambda}) \right] \delta_{\bar{C}\bar{D}}^{AB} + (\ell\bar{\ell})^{-1} \left[\bar{\Lambda}^{AB}\Lambda_{CD} + 4\delta_{[C}^{[A}(\Lambda\bar{\Lambda})_{D]}^{B]} \right] . \end{aligned} \quad (2.14)$$

It is difficult to invert this metric directly. Instead we make use of our knowledge of the embedding space metric in the following procedure. Any tangent space vector can be thought of $so(10)$ -covariantly as being projected by the projection operator[†]

$$P^\alpha{}_\beta = \delta^\alpha{}_\beta - \frac{1}{2}(\lambda\bar{\lambda})^{-1}(\gamma^a\bar{\lambda})^\alpha(\gamma_a\lambda)_\beta = (\lambda\bar{\lambda})^{-1} \left[-\frac{1}{4}\lambda^\alpha\bar{\lambda}_\beta + \frac{1}{8}(\gamma^{ab}\lambda)^\alpha(\gamma_{ab}\bar{\lambda})_\beta \right] , \quad (2.15)$$

acting as the identity on any spinor v^α with $(v\gamma^a\lambda) = 0$. A conjugate tangent vector is projected by $\bar{P} = P^t$. When the embedding space metric is the flat one, $P^{\alpha\bar{\beta}}$ can be thought of as the metric. Its inverse on the tangent directions is the metric P itself. The inverse metric in some coordinate system is obtained from this inverse metric. Specifically, in the

[†] This projection also occurs in ref. [29]. With $\bar{\lambda}$ replaced by a constant pure spinor, such operators appear also *e.g.* in ref. [20].

$SU(5)$ -covariant coordinate system, we take cotangent vectors decomposed as in eq. (2.4), but without 4-form (or 1-form): $V = V_3 + V_5$, $\bar{V} = \bar{V}_0 + \bar{V}_2$. Contracting these cotangent vectors with the covariant inverse metric gives the inverse metric in the coordinate system, which reads explicitly:

$$\begin{aligned}
(h^{-1})_{00} &= 1 + \frac{1}{4}(\lambda\bar{\lambda})^{-1}(\ell\bar{\ell})^{-1}X , \\
(h^{-1})_{AB,\bar{0}} &= (\lambda\bar{\lambda})^{-1}(\ell\bar{\ell})^{-1}\bar{\ell} \left[(\bar{\Lambda}\Lambda\bar{\Lambda})_{AB} - \frac{1}{2}\Lambda_{AB}\text{tr}(\Lambda\bar{\Lambda}) \right] , \\
(h^{-1})_{AB,\bar{C}\bar{D}} &= 2 \left[1 + \frac{1}{2}(\lambda\bar{\lambda})^{-1}\text{tr}(\Lambda\bar{\Lambda}) + \frac{1}{2}(\lambda\bar{\lambda})^{-1}(\ell\bar{\ell})^{-1}X \right] \delta_{AB}^{CD} \\
&\quad - (\lambda\bar{\lambda})^{-1} \left[\Lambda_{AB}\bar{\Lambda}^{CD} + 4\delta_{[A}^{[C}(\Lambda\bar{\Lambda})_{B]}^{D]} \right] \\
&\quad - 4(\lambda\bar{\lambda})^{-1}(\ell\bar{\ell})^{-1}\delta_{[A}^{[C} \left[(\Lambda\bar{\Lambda}\Lambda\bar{\Lambda})_{B]}^{D]} - \frac{1}{2}(\Lambda\bar{\Lambda})_{B]}^{D]} \text{tr}(\Lambda\bar{\Lambda}) \right] .
\end{aligned} \tag{2.16}$$

Already in this expression we have used some identities from the Appendix to simplify the expressions. The expression $(\lambda\bar{\lambda})$ should here always be understood as the expression in eq. (2.11). We have verified that the metric and inverse metric of eqs. (2.14) and (2.16) satisfy $hh^{-1} = 1$, but this is a lengthy calculation involving the relations in the Appendix.

We now turn to the actual metric on the pure spinor space allowing for the holomorphic volume form Ω . The covariant form (2.13) can be understood as a metric

$$G_{\alpha\bar{\beta}} = (\lambda\bar{\lambda})^{-3/11} \left[P_{\alpha\bar{\beta}} - \frac{3}{11}(\lambda\bar{\lambda})^{-1}\bar{\lambda}_{\alpha}\lambda_{\bar{\beta}} \right] . \tag{2.17}$$

The last term is automatically tangent. Its inverse \tilde{G} on tangent space (by which we mean $G\tilde{G} = P$) is

$$\tilde{G}^{\alpha\bar{\beta}} = (\lambda\bar{\lambda})^{3/11} \left[P^{\alpha\bar{\beta}} + \frac{3}{8}(\lambda\bar{\lambda})^{-1}\lambda^{\alpha}\bar{\lambda}^{\bar{\beta}} \right] = \frac{1}{8}(\lambda\bar{\lambda})^{-8/11} \left[\lambda^{\alpha}\bar{\lambda}^{\bar{\beta}} + (\gamma^{ab}\lambda)^{\alpha}(\gamma_{ab}\bar{\lambda})^{\bar{\beta}} \right] . \tag{2.18}$$

In the $SU(5)$ coordinate system, using the same procedure as before to form the metric and its inverse, this amounts to

$$\begin{aligned}
g &= (\lambda\bar{\lambda})^{-3/11} \left[h - \frac{3}{11}(\lambda\bar{\lambda})^{-1}(\beta \otimes \bar{\beta} + \bar{\beta} \otimes \beta) \right] , \\
g^{-1} &= (\lambda\bar{\lambda})^{3/11} \left[h^{-1} + \frac{3}{8}(\lambda\bar{\lambda})^{-1}(\gamma \otimes \bar{\gamma} + \bar{\gamma} \otimes \gamma) \right] ,
\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}
\beta^0 &= \left[1 + \frac{1}{4}(\ell\bar{\ell})^{-2}X \right] \bar{\ell} , \\
\beta^{AB} &= \bar{\Lambda}^{AB} + (\ell\bar{\ell})^{-1} \left[(\bar{\Lambda}\Lambda\bar{\Lambda})^{AB} - \frac{1}{2}\bar{\Lambda}^{AB}\text{tr}(\Lambda\bar{\Lambda}) \right] , \\
\gamma_0 &= \ell , \\
\gamma_{AB} &= \Lambda_{AB} .
\end{aligned} \tag{2.20}$$

It is easily checked that $\bar{\beta} = h\gamma$ and that $\beta \cdot \gamma = (\lambda\bar{\lambda})$, ensuring that g^{-1} is the inverse of g .

2.4. COVARIANT EXPRESSIONS FOR OPERATORS

The inverse metric is needed for the construction of metric dependent differential operators, *e.g.* the Laplacian. The Laplacian on a Kähler space simplifies, in that it can be given in terms of the Dolbeault operator $\bar{\partial}$ as $\Delta_{\bar{\partial}} = \{\bar{\partial}, \bar{\partial}^*\}$, where $\bar{\partial}^* = \star \bar{\partial} \star$ is the adjoint operator to $\bar{\partial}$. $\Delta_{\bar{\partial}}$ is proportional to the ordinary Laplacian, $2\Delta_{\bar{\partial}} = \Delta_d = \{d, d^*\}$.

In ref. [20], regularisation of higher loop integrals was performed by using an operator that was argued to have some close relationship with the Laplacian (although we suspect that if there is an identity it may well be with the Laplacian corresponding to the metric h). The b operator was shown to become regular when modified as $b' = e^{t\{Q, \chi\}} b e^{-t\{Q, \chi\}}$, with the "regulating fermion" χ , roughly speaking, being proportional to $\bar{\partial}^*$.

We have $\{Q, \bar{\partial}^*\} = \Delta_{\bar{\partial}} + \dots$. The operator $\bar{\partial}^*$ should be formed as the divergence $g^{m\bar{n}} \bar{\iota}_{\bar{n}} \partial_m$. Here the contraction may be represented as a field s and the derivative as a w . In this picture, these spinor operators must come in some gauge invariant combination. It is clear from above how this is achieved. One should use the $so(10)$ -covariant form \tilde{G} of the inverse metric:

$$\bar{\partial}^* = \tilde{G}^{\alpha\bar{\beta}} s_{\bar{\beta}} w_{\alpha} . \quad (2.21)$$

Consequently,

$$\{Q, \bar{\partial}^*\} = \Delta_{\bar{\partial}} - \tilde{G}^{\alpha\bar{\beta}} s_{\bar{\beta}} D_{\alpha} . \quad (2.22)$$

This also gives an $so(10)$ -covariant prescription for the Laplacian. As an example, the Laplacian acting on a $(0, 0)$ -form ϕ is

$$\Delta_{\bar{\partial}} \phi = \bar{\partial}^* \bar{\partial} \phi = \tilde{G}^{\bar{\beta}\alpha} w_{\alpha} \bar{w}_{\bar{\beta}} \phi = \frac{1}{8} (\lambda\bar{\lambda})^{-8/11} (N\bar{N} + N^{ab} \bar{N}_{ab}) \phi , \quad (2.23)$$

where N and N^{ab} are the well defined quantities $N = (\lambda w)$, $N^{ab} = (\lambda \gamma^{ab} w)$. In principle, one could think of other constructions, *e.g.* using the matrix P instead of \tilde{G} . Such operators will however be less natural, and will not have a geometric interpretation on pure spinor space.

We have not yet checked whether or not such a geometric regularisation will yield a regular b operator, but find it plausible considering the work in ref. [20]. We comment more on related issues in the concluding section.

3. FURTHER DIRECTIONS

It would be interesting if a more geometric approach can help in defining regular operators (in particular, the b operator) on pure spinor space. Any gauge fixing operator b should have the property $\{Q, b\} \propto p^2$, but this does not uniquely fix b . Instead there is a gauge degree of freedom amounting to shifting b with something Q -exact, and this is the freedom one uses in regularisation. One possibility might lie in using the localisation obtained by letting the parameter t in the exponent tend to infinity, which would mean only dealing with zero eigenstates of $\Delta_{\bar{\partial}} - \tilde{G}^{\alpha\bar{\beta}} s_{\bar{\beta}} D_{\alpha}$. This would be a quite natural Q -exact generalisation of choosing harmonic forms as representatives for cohomology. We would like to examine whether this or some similar geometrically motivated procedure can provide a good regularisation.

Another issue is generalisation to $D = 11$. In order to investigate properties of amplitudes in dimensional reductions of $D = 11$ supergravity, we believe that the covariant action of refs. [15,16] will be the best starting point. Some work has been done in a first-quantised formalism [30,31]. The geometry of $D = 11$ pure spinor space is more complicated than in $D = 10$ [32,6,15,16,33], mainly due to the existence of two independent $so(11)$ scalars, $(\lambda\bar{\lambda})$ and $(\lambda\gamma^{ab}\lambda)(\bar{\lambda}\gamma_{ab}\bar{\lambda})$. The Kähler potential will depend only on these. The b operator has been constructed [34], but further regularisation will certainly be needed.

APPENDIX A: SOME MATRIX IDENTITIES

A Cayley–Hamilton relation for matrix products of Λ ’s and $\bar{\Lambda}$ ’s, obtained from antisymmetrisation in 6 indices:

$$0 = \Lambda\bar{\Lambda}\Lambda\bar{\Lambda}\Lambda - \frac{1}{2}\Lambda\bar{\Lambda}\Lambda\text{tr}(\Lambda\bar{\Lambda}) - \frac{1}{4}\Lambda X, \quad (\text{A.1})$$

where $X = \text{tr}(\Lambda\bar{\Lambda})^2 - \frac{1}{2}(\text{tr}(\Lambda\bar{\Lambda}))^2$. Another useful relation*, necessary for checking the form of the inverse metric, is

$$\begin{aligned} 0 = & \Lambda_{AB}(\bar{\Lambda}\Lambda\bar{\Lambda})^{CD} + (\Lambda\bar{\Lambda}\Lambda)_{AB}\bar{\Lambda}^{CD} + 2(\Lambda\bar{\Lambda})_{[A}{}^C(\Lambda\bar{\Lambda})_{B]}{}^D - \frac{1}{2}\Lambda_{AB}\bar{\Lambda}^{CD}\text{tr}(\Lambda\bar{\Lambda}) \\ & + 4\delta_{[A}{}^{[C}[(\Lambda\bar{\Lambda}\Lambda\bar{\Lambda})_{B]}{}^{D]} - \frac{1}{2}(\Lambda\bar{\Lambda})_{B]}{}^{D]}\text{tr}(\Lambda\bar{\Lambda})] - \frac{1}{2}\delta_{AB}^{CD}X, \end{aligned} \quad (\text{A.2})$$

* Note that the existence of some such relation can be deduced from the fact that the tensor product of the symmetric tensor product of 2 Λ ’s (in (0100)) and the symmetric tensor product of 2 $\bar{\Lambda}$ ’s (in (0010)) only contains 3 structures in (0110), and this module is contained in the 4 terms of the first line of the identity. Thus there must be $4 - 3 = 1$ identity.

which can also be obtained by cycling in 6 indices. It is straightforward to check that it is consistent with contraction by δ_D^B (by direct calculation) and with Λ_{CD} or $\bar{\Lambda}^{AB}$ (thanks to eq. (A.1)). As a consequence of eqs. (A.2) and (A.1) one also gets

$$\begin{aligned} 0 = & (\Lambda\bar{\Lambda})_{AB}(\bar{\Lambda}\Lambda\bar{\Lambda})^{CD} + 4(\Lambda\bar{\Lambda})_{[A}^{[C}(\Lambda\bar{\Lambda}\Lambda\bar{\Lambda})_{B]}^{D]} \\ & - (\Lambda\bar{\Lambda})_{[A}^{[C}(\Lambda\bar{\Lambda})_{B]}^{D]}\text{tr}(\Lambda\bar{\Lambda}) + \frac{1}{4}\Lambda_{AB}\bar{\Lambda}^{CD}X . \end{aligned} \quad (\text{A.3})$$

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